

Tues, 1 April: First recap Cuttingham setup

$\Im M =$

$\Im \nu =$

dispersion integrals

200

so starting off the contributions will be finite

In our case, we have the functions T_i defined for each

$$T_i(\nu, Q^2)$$

We are going to do a fixed- Q^2 dispersion integral

integrated over ν from 0 to ∞ $\int_0^\infty d\nu = (\text{exp})$

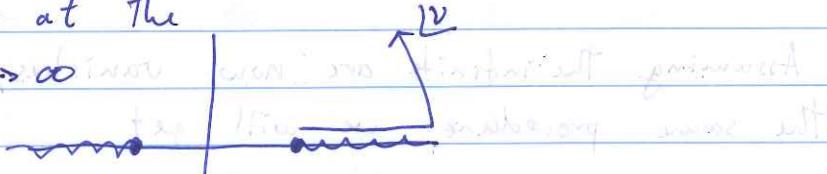
hold Q^2 fixed, take $\nu \rightarrow$ complex

$\text{cusp} \rightarrow \infty$

Note: $Q^2 \rightarrow \infty$ is where we know how to analytically continue and connect w/p QCD. It is this limit where we already know how to renormalize the theory

For these dispersion integrals, what we are interested in is the behavior at the

infinite arc, $\Im \nu \rightarrow \infty$



This is the "Regge Limit".

There is a wealth of information about these functions in the Regge limit.

What is known empirically is T_1, t_1 satisfy unsubtracted dispersion relations

T_1, t_1 require a subtracted dispersion integral

To connect to the last couple lectures, you can show

$$(2\Im M T_1(\nu, Q^2) = 2\pi F_1(\nu, Q^2))$$

$$2\Im M T_2(\nu, Q^2) = 2\pi \frac{M}{\nu} F_2(\nu, Q^2)$$

$$2\Im M t_1(\nu, Q^2) = \frac{2\pi M \nu}{Q^4} \left[2x F_1(x, Q^2) - F_2(x, Q^2) \right], \quad x = \frac{Q^2}{2M\nu}$$

multiplied by t_1 gives the sum of present problems all

and this is our Callan-Gross relation, which we stated received known QCD corrections

$$2x F_1(x, Q^2) - F_2(x, Q^2) = -\frac{32}{9} \frac{\alpha_s(Q^2)}{4\pi} F_2(x, Q^2)$$

It is known, in the Regge Limit ($Q^2 = \text{fixed}$, $v \rightarrow \infty$)

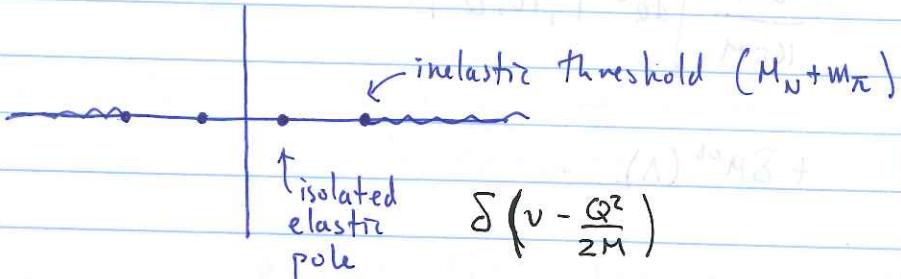
$$\lim_{x \rightarrow 0} F_2^N(x, Q^2) \propto x^\sigma \Rightarrow \text{Im } t_1 \propto v \quad (\text{Pomeron})$$

$$\lim_{x \rightarrow 0} F_2^{P-n}(x, Q^2) \propto \sqrt{x} \Rightarrow \text{Im } t_1^{P-n} \propto \sqrt{v} \quad (\text{A}_2\text{-Regge trajectory})$$

$$\Rightarrow T_1(iv, Q^2) = T_1(0, Q^2) - \frac{v^2}{2\pi} \int_{v_t}^{\infty} dv' \frac{2v'}{v'^2(v'^2 + v^2)} 2 \text{Im } T_1(v'_+, Q^2)$$

$$T_2(iv, Q^2) = \frac{1}{2\pi} \int_{v_t}^{\infty} dv' \frac{2v'}{v'^2 + v^2} 2 \text{Im } T_2(v'_+, Q^2)$$

So, what does the analytic structure of these functions look like?



So we can separate the elastic contributions and easily evaluate it, and be left with a double integral for the inelastic region

$$\delta M = -\frac{\alpha_{\text{q.s.}}}{4\pi^2} \int_0^{\infty} dQ^2 \int_{-Q}^Q dv \left[\frac{3T_1(iv, Q^2)}{2M} - \left(1 - \frac{v^2}{Q^2}\right) \frac{T_2(iv, Q^2)}{2M} \right]$$

$$\Delta M = -\frac{\alpha_{\text{f.s.}}}{4\pi^2} \int_0^\infty dQ^2 \int_{-Q}^Q \frac{\sqrt{Q^2 - v^2}}{Q^2} \left[\frac{3T_1(0, Q^2)}{2M} - \frac{v^2}{4M\pi} \int_{v_t}^\infty du \frac{2u}{u^2(u^2 + v^2)} \frac{2\text{Im}T_1}{2M} \right. \\ \left. - \left(1 - \frac{v^2}{Q^2}\right) \frac{1}{2\pi} \int_{v_t}^\infty du \frac{2u}{u^2 + v^2} \frac{2\text{Im}T_2}{2M} \right]$$

$$2\text{Im}T_1^{el} = 2M \cdot 2\pi \delta(v - \frac{Q^2}{2M}) \frac{Q^2}{4M^2} G_M^2(Q^2)$$

So the $\int dv$ integral can be performed easily for the elastic terms.

$$\Delta M = \frac{\alpha}{\pi} \int_0^\Lambda dQ \left\{ \frac{3}{2} \frac{\sqrt{\tau^{el}} G_M^2}{1 + \tau^{el}} + \frac{G_E^2 - 2\tau^{el} G_M^2}{1 + \tau^{el}} \left[(1 + \tau^{el})^{3/2} - (\tau^{el})^{3/2} - \frac{3}{2} \sqrt{\tau^{el}} \right] \right\} \\ + \frac{\alpha}{4\pi M} \int_0^\Lambda \frac{dQ^2}{Q} \int_{W_{th}^2}^\infty dW^2 \left\{ \frac{3F_1(v, Q^2)}{M} \left[\frac{\tau^{3/2} - \sqrt{1 + \tau}}{\tau} + \frac{1}{2}\sqrt{\tau} \right] \right. \\ \left. + \frac{F_2(v, Q^2)}{v} \left[(1 + \tau)^{3/2} - \tau^{3/2} - \frac{3}{2}\sqrt{\tau} \right] \right\}$$

$$- \frac{3\alpha}{16\pi M} \int_0^\Lambda dQ^2 T_1(0, Q^2)$$

$$+ \Delta M^{el.}(\Lambda)$$

$$\tau^{el} = \frac{Q^2}{4M^2}, \quad \tau = \frac{v^2}{Q^2}, \quad W^2 = M^2 + 2Mv - Q^2, \quad W_{th} = M + m_\pi$$

$$[(M, m_\pi)T(\tau^2 - 1) - (m_\pi)T\tau] \cdot \tau^2 \cdot \frac{1}{\tau^2} = M\bar{v}$$

What can we say about this subtraction function?

$$\lim_{Q^2 \rightarrow \infty} T_1(0, Q^2) \propto \frac{1}{Q^2}$$

This can be shown with the perturbative QCD.

If we just believe the theory is renormalizable, we can see this must be the behavior, so that

$$\int dQ^2 T_1(0, Q^2) \sim \ln Q^2 \text{ in the U.V.}$$

This is the log divergence arising from the quark QED self-energy (just like the e^-)

The counter term must be proportional to the quark masses as well (if $m_q = 0$, then the theory has no U.V. divergence - X -symmetry)

Also, the c.t. must have a log which exactly cancels the log divergence from $\int dQ^2 T_1(0, Q^2)$

After cancelling the \ln , there is some finite residual stuff, which can be estimated

$$\tilde{\delta M}^{ct} \approx -\frac{3\alpha}{4\pi} \ln\left(\frac{\Lambda^2}{\Lambda_0^2}\right) \frac{e_u^2 m_u - e_d^2 m_d}{\delta} \frac{\langle p | \delta(\bar{u}u - \bar{d}d) | p \rangle}{\langle p | \bar{u}u + \bar{d}d | p \rangle} \cdot \frac{\langle p | \hat{m}(\bar{u}u + \bar{d}d) | p \rangle}{2M}$$

$$m_u = \hat{m} - \delta, \quad m_d = \hat{m} + \delta$$

$$(e_u^2 - e_d^2)\hat{m} - (e_d^2 + e_u^2)\delta = \frac{1}{q}(3\hat{m} - 5\delta)$$

$$\sigma_{\bar{u}u} \approx 45 \text{ MeV}$$

$$\frac{\langle p | \bar{u}u - \bar{d}d | p \rangle}{\langle p | \bar{u}u + \bar{d}d | p \rangle} \leq \frac{2-1}{2+1} = \frac{1}{3}$$

At low energy, you can also bound this from

$$\lim_{Q^2 \rightarrow 0} T_1(0, Q^2) = 2 \kappa (2 + \kappa)$$

$$-Q^2 \left\{ \frac{2}{3} \left[(1+\kappa)^2 r_M^2 - r_E^2 \right] + \frac{\kappa}{M^2} - 2M \frac{\beta_M}{\alpha} \right\}$$

$$r_{E,M}^2 = 6 \lim_{Q^2 \rightarrow 0} \frac{\partial}{\partial Q^2} G_{E,M}(Q^2)$$

$\kappa = F_2(0)$ = anomalous magnetic moment

β_M = magnetic polarizability

Aside: this same function, $T_1(0, Q^2)$ appears in
determination of proton size through (μ)-Hydrogen

$$\overbrace{\qquad\qquad\qquad}^{e, \mu} \overbrace{\qquad\qquad\qquad}^{\text{similar integral, but}} P \qquad \text{extra fermion propagator}$$

shift due to suppression of terms with γ^5

$$T_1(0, Q^2) = \frac{1}{2} (1 + \kappa)^2 r_M^2 + \frac{1}{2} M^2 \gamma^5 \frac{1}{Q^2} \gamma^5$$

$$2M^2 \gamma^5 = 2 \left(\frac{1}{Q^2} + \frac{1}{M^2} \right) \gamma^5$$

$$\frac{1}{Q^2} + \frac{1}{M^2} = \frac{1}{Q^2} \left(1 + \frac{M^2}{Q^2} \right)$$